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Some inequalities and asymptotic formulas for eigenvalues on Riemannian manifolds

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ABSTRACT

In this paper, we establish sharp inequalities for four kinds of classical eigenvalues in bounded domains of Riemannian manifolds. We also obtain the Weyl-type asymptotic formulas for the eigenvalues of the buckling and clamped plate problems in bounded domains of Riemannian manifolds. In addition, we give a negative answer to the Payne conjecture for the one-dimensional case.

1. Introduction

Let (M, g) be an n-dimensional oriented Riemannian manifold, and let $\Omega \subset M$ be a bounded domain with C^2-smooth boundary $\partial\Omega$. We consider the following classical eigenvalue problems:

$$\begin{cases} \triangle_g u + \mu u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

$$\begin{cases} \triangle_g u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

$$\begin{cases} \triangle_g^2 u - \Gamma^2 u = 0 & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

$$\begin{cases} \triangle_g^2 u + \Lambda \triangle_g u = 0 & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

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where ν denotes the outward unit normal vector to $\partial\Omega$, and $\triangle_g$ is the Laplace–Beltrami operator defined in local coordinates by the expression,

$$\triangle_g = \frac{1}{\sqrt{|g|}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{|g|}g^{ij}\frac{\partial}{\partial x_j}\right).$$

Here $|g| := \det(g_{ij})$ is the determinant of the metric tensor, and g^{ij} are the components of the inverse of the metric tensor g.

(1.1) is the Neumann problem (see [9]); (1.2) is the Dirichlet problem (see [9] or [11]); (1.3) occurs in the treatment of the vibration problem for a clamped plate (see [11] and [40]), and (1.4) is the well-known buckling problem for a clamped plate (see [9], [11], [33], [34] or [40]). In each of these cases, the spectrum is discrete and we arrange the eigenvalues in non-decreasing order (repeated according to multiplicity)

$$0 = \mu_1 < \mu_2 \leqslant \cdots \leqslant \mu_k \leqslant \cdots;$$
$$0 < \lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_k \leqslant \cdots;$$
$$0 < \Gamma_1^2 \leqslant \Gamma_2^2 \leqslant \cdots \leqslant \Gamma_k^2 \leqslant \cdots;$$
$$0 < \Lambda_1 \leqslant \Lambda_2 \leqslant \cdots \leqslant \Lambda_k \leqslant \cdots.$$

The corresponding eigenfunctions are expressed as $v_1, v_2, v_3, \ldots$; $u_1, u_2, u_3, \ldots$; $U_1, U_2, U_3, \ldots$; and $W_1, W_2, W_3, \ldots$.

For any bounded domain Ω in (M, g), the variational formulation of the Neumann and Dirichlet eigenvalue problems (in terms of Rayleigh quotients, cf. Section VI.1 of [11]) immediately implies the inequalities

$$\mu_k \leqslant \lambda_k, \quad k = 1, 2, 3, \ldots.$$

Moreover, Pólya [39] proved in 1952 that for $\Omega \subset \mathbb{R}^2$,

$$\mu_2 < \lambda_1, \tag{1.5}$$

answering a question of Kornhauser and Stakgold [25]. In the case that Ω is a bounded convex domain $\Omega \subset \mathbb{R}^2$ with a piecewise C^2-smooth boundary, Payne [33] showed that

$$\mu_{k+2} < \lambda_k, \quad k = 1, 2, 3, \ldots. \tag{1.6}$$

Levine and Weinberger [27] proved that

$$\mu_{k+n} < \lambda_k, \quad k = 1, 2, 3, \ldots \tag{1.7}$$

for smooth bounded convex domains $\Omega \subset \mathbb{R}^n$ (cf. [5]), as well as

$$\mu_{k+m} \leqslant \lambda_k, \quad k = 1, 2, 3, \ldots;\ 1 \leqslant m \leqslant n \tag{1.8}$$

for arbitrary bounded convex domains. In 1991, Friedlander [15] proved that

$$\mu_{k+1} \leqslant \lambda_k, \quad k = 1, 2, 3, \ldots \tag{1.9}$$

when $\Omega \subset \mathbb{R}^n$ is a bounded domain with a C^1-smooth boundary $\partial\Omega$. We also refer to Mazzeo [28] for an extension to certain smooth manifolds, and to Ashbaugh and Levine [4] and Hsu and Wang [20] for the case of subdomains of the n-sphere $\mathbb{S}^n$ with a smooth boundary and nonnegative mean curvature. Finally, in 2004 Filonov [14] proved strict inequality

$$\mu_{k+1} < \lambda_k \quad (k = 1, 2, 3, \ldots), \tag{1.10}$$

when $\Omega \subset \mathbb{R}^n$ is a domain with finite volume, and with the embedding $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ compact.

In regard to the vibration problem of a clamped plate, Pólya in [38] obtained that

$$\lambda_k \leqslant \Gamma_k, \quad k = 1, 2, 3, \ldots$$

for any bounded domain in $\mathbb{R}^2$. This result had actually been improved to be

$$\lambda_k < \Gamma_k, \quad k = 1, 2, 3, \ldots \tag{1.11}$$

for bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary by Weinstein [48] (which was referred as Weinstein's inequality in [49]). In [3], Ashbaugh and Laugesen obtained the inequalities

$$\lambda_1^2 \leqslant \lambda_1\lambda_2 \leqslant \Lambda_1\lambda_1 \leqslant \Gamma_1^2 \leqslant \Lambda_1^2. \tag{1.12}$$

Concerning the buckling problem of a clamped plate, in 1937 Weinstein [48] proved the following strictly inequality

$$\lambda_k < \Lambda_k, \quad k = 1, 2, 3, \ldots \tag{1.13}$$

for any bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary. Payne [33] in 1955 proved that for any bounded domain $\Omega \subset \mathbb{R}^2$ with C^2-smooth boundary

$$\lambda_2 \leqslant \Lambda_1, \tag{1.14}$$

solving a conjecture of Weinstein. In [33], Payne further made the conjecture:

$$\lambda_{k+1} \leqslant \Lambda_k, \quad \text{for } k = 1, 2, 3, \ldots. \tag{1.15}$$

Note that this remarkable conjecture remains open in $\mathbb{R}^n$ ($n \geqslant 2$).

The first purpose of this paper is to prove:

Theorem 1.1. *Let (M, g) be an n-dimensional oriented Riemannian manifold ($n \geqslant 2$), and let $\Omega \subset M$ be a bounded domain with C^2-smooth boundary. Then*

$$\mu_k < \lambda_k < \Gamma_k < \Lambda_k \quad \text{for } k = 1, 2, 3, \ldots, \tag{1.16}$$

where μ_k, λ_k, Γ_k^2 and Λ_k are the k-th eigenvalues of the Neumann, Dirichlet, clamped plate and buckling problems for the domain Ω, respectively.

We also show by some examples that in the Riemannian manifold setting, (1.16) are the best possible inequalities for these classical eigenvalue problems (see Remark 3.1).

H. Weyl in 1912, was the first to establish asymptotic formulas in $\mathbb{R}^n$ for the Dirichlet and Neumann eigenvalues (see [50] or [51]):

$$\lambda_k \sim (2\pi)^2 \left(\frac{k}{\omega_n(vol(\Omega))} \right)^{2/n}, \quad k \to \infty, \tag{1.17}$$

$$\mu_k \sim (2\pi)^2 \left(\frac{k}{\omega_n(vol(\Omega))} \right)^{2/n}, \quad k \to \infty, \tag{1.18}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$, and $vol(\Omega)$ is the n-dimensional Lebesgue volume of Ω ($\sim$ means the ratio of the RHS to the LHS approaches 1 as $k \to \infty$). In the case of two-dimensional Euclidean space, Pleijel [37] in 1950 gave the asymptotic formula for the eigenvalues of a clamped plate based on a Carleman's method in [6] and [7]. In 1980, Grubb [18] (also see [2]) gave an asymptotic formula for the buckling problem in $\mathbb{R}^n$. In 1967, McKean and Singer [29] generalized Weyl's asymptotic formulas to a bounded domain of a Riemannian manifold by investigating asymptotic expansion of the trace of heat operator.

The second purpose of the paper is to give a negative answer to the Payne conjecture for the one-dimensional case (see Section 4). Furthermore, we establish the asymptotic formulas for the eigenvalues of the buckling and clamped plate problems in a bounded domain of a Riemannian manifold. We have the following:

Theorem 1.2. *Let (M, g) be an n-dimensional Riemannian manifold, and let $\Omega \subset M$ be a bounded domain with C^2-smooth boundary. Then*

$$\Lambda_k \sim (2\pi)^2 \left(\frac{k}{\omega_n(vol(\Omega))} \right)^{2/n}, \quad \text{as } k \to +\infty, \tag{1.19}$$

$$\Gamma_k \sim (2\pi)^2 \left(\frac{k}{\omega_n(vol(\Omega))} \right)^{2/n}, \quad \text{as } k \to +\infty, \tag{1.20}$$

where $vol(\Omega)$ is the volume of Ω.

McKean–Singer–Weyl's asymptotic formulas and Theorem 1.2 show that for general bounded domain with C^2-smooth boundary in a Riemannian manifold, the four kinds of classical quantities μ_k, λ_k, Γ_k and Λ_k have the same (leading term) asymptotic behavior as $k \to +\infty$. In other words, the formula (1.16) is also the best possible inequalities in the asymptotic sense.

The proof of Theorem 1.1 uses a key result (Lemma 2.1), which generalizes the classical Holmgren's uniqueness theorem (see [42]) to the Rienannian manifold, and a technique of [14] by which Filonov proved the inequalities (1.10). In order to prove Theorem 1.2, we first consider the case of the Euclidean space and then obtain the version in Riemannian manifold by applying metric expansion in normal coordinates system. The main method is to approximate Ω by a union of subdomains whose boundary are piecewise smooth that has been suitably contracted. We thus get a lower estimate for the counting function of the buckling eigenvalues if these subdomains are open, disjoint and lie inside Ω. In our discussion, the Bochner–Lichnerowicz–Weitzenböck formula plays an important role. On the other hand, an upper estimate had been given in [29] (see also p. 441 of [11]) by investigating the Neumann and Dirichlet eigenvalue problems. Thus the desired result is proved.

2. Some lemmas

The following several lemmas will be needed.

Lemma 2.1. *Let Ω be a bounded domain with C^2-smooth boundary in an n-dimensional Riemannian manifold (M, g), and let $0 \neq u \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ be a solution of (1.2). Then $\frac{\partial u}{\partial \nu}|_{\partial\Omega}$ does not vanish identically on $\partial\Omega$.*

Proof. Let $F(x,\xi)$ be a fundamental solution for the Helmholtz operator $\triangle_g + \lambda$ on M (i.e., $F(x,\xi)$ satisfying

$$(\triangle_g + \lambda)F(x,\xi) = \delta_x(\xi), \tag{2.1}$$

where $\triangle_g$ denotes the Laplace operator taken with respect to the variables ξ, and $\delta_x(\xi)$ is the Dirac δ-function. More precisely, $(\triangle_g + \lambda)F(x,\xi) = 0$ with respect to $\xi \neq x$ for any fixed x). For $x \in M$, we choose the normal coordinates centered at x. Since $F(x,\xi)$ is singular at $\xi = x$ we cut out from Ω a geodesic ball $B(x,\epsilon)$ contained in Ω with center x, radius $\epsilon > 0$. From $u \in W_0^{1,2}(\Omega)$, we find by the same argument as in Corollary 6.2.43 of [19] that $u = 0$ on $\partial\Omega$. Since $\triangle_g F(x,\xi) + \lambda F(x,\xi) = 0$ in $\Omega \setminus B(x,\epsilon)$, by Green's formula we obtain

$$\begin{aligned}
0 &= \int_{\Omega\setminus B(x,\epsilon)} u(\xi)\big(\triangle_g F(x,\xi) + \lambda F(x,\xi)\big)\, dV_g(\xi) \\
&= \int_{\Omega\setminus B(x,\epsilon)} \big(\triangle_g u(\xi)\big) F(x,\xi)\, dV_g(\xi) + \int_{\partial(\Omega\setminus B(x,\epsilon))} u(\xi)\frac{\partial F(x,\xi)}{\partial \nu_\xi}\, dS_g(\xi) \\
&\quad - \int_{\partial(\Omega\setminus B(x,\epsilon))} F(x,\xi)\frac{\partial u(\xi)}{\partial \nu_\xi}\, dS_g(\xi) + \lambda \int_{\Omega\setminus B(x,\epsilon)} u(\xi) F(x,\xi)\, dV_g(\xi) \\
&= \int_{\partial(\Omega\setminus B(x,\epsilon))} u(\xi)\frac{\partial F(x,\xi)}{\partial \nu_\xi}\, dS_g(\xi) - \int_{\partial(\Omega\setminus B(x,\epsilon))} F(x,\xi)\frac{\partial u(\xi)}{\partial \nu_\xi}\, dS_g(\xi) \\
&= -\int_{\partial B(x,\epsilon)} u(\xi)\frac{\partial F(x,\xi)}{\partial \nu_\xi}\, dS_g(\xi) - \int_{\partial\Omega} F(x,\xi)\frac{\partial u(\xi)}{\partial \nu_\xi}\, dS_g(\xi) + \int_{\partial B(x,\epsilon)} F(x,\xi)\frac{\partial u(\xi)}{\partial \nu_\xi}\, dS_g(\xi),
\end{aligned}$$

where $dS_g(\xi)$ denotes the $(n-1)$-dimensional volume element, and $\frac{\partial}{\partial \nu_\xi}$ denotes the derivative in the direction of the outward unit normal vector ν_ξ at ξ. We now wish to evaluate the limits of the individual integrals in this formula for $\epsilon \to 0$. On $\partial B(x,\epsilon)$, we have $F(x,\xi) = F_1(\epsilon) + O(\epsilon)$ since we have used the normal coordinates. From proof of Theorem 9.4 of [31], we know that for $n \geqslant 2$,

$$F_1(z) = F_0(z)\big[1 + f(z)\big] + h(z) \quad \text{as } |z| \to 0, \tag{2.2}$$

where

$$F_0(z) = \begin{cases} \frac{|z|^{2-n}}{n(2-n)\omega_n} & \text{for } n > 2, \\ \frac{1}{2\pi}\log|z| & \text{for } n = 2, \end{cases} \tag{2.3}$$

$$f(z) = O\big(|z|^2\big)$$

and

$$h(z) = \begin{cases} \text{const} + O(|z|^2) & \text{for } n = 2, \\ 0 & \text{for odd } n > 2, \\ \text{const} \times \log(\sqrt{\lambda}|z|/2)[1 + O(|z|^2)] & \text{for even } n > 2; \end{cases} \tag{2.4}$$

here ω_n, as before, denotes the volume of the unit ball in $\mathbb{R}^n$, and the $O(|z|^2)$ terms are analytic functions of $|z|^2$.

Under the normal coordinates,

$$\xi = q(\epsilon, \eta) = \exp_x \epsilon\eta,$$

$\epsilon \geqslant 0$, $\eta \in \mathfrak{S}_x = \{\eta \in M_x \mid |\eta| = 1\}$, about x. As discussed in Section III.1 of [9], the volume element dV_g is given by

$$dV_g\big(q(\epsilon, \eta)\big) = \sqrt{\big|g(\epsilon, \eta)\big|}\, d\epsilon\, d\mu_x(\eta),$$

where $d\mu_x$ is the standard $(n-1)$-measure on $\mathfrak{S}_x$; and the $(n-1)$-dimensional volume element of $\partial B(x, r)$ is given by

$$dS_g\big(q(\epsilon, \eta)\big) = \sqrt{g(\epsilon, \eta)}\, d\mu_x(\eta).$$

The discussion of Sections III.1 and XII.8 of [9] shows that

$$\lim_{\epsilon \to 0} \frac{\sqrt{|g(\epsilon, \eta)|}}{\epsilon^{n-1}} = 1.$$

Since $u \in W^{2,2}(\Omega)$, by applying local regularity of elliptic equations (see, for example, Theorem A.2.1 of [23]) repeatedly, we get that $u \in W^{j,2}(\overline{B(x, \epsilon)})$ for all $j = 1, 2, 3, \ldots$, which implies $u \in C^\infty(\overline{B(x, \epsilon)})$. It follows from (2.2) and (2.3) that for $\epsilon \to 0$,

$$\left| \int_{\partial B(x,\epsilon)} F(x, \xi) \frac{\partial u(\xi)}{\partial \nu_\xi}\, dS_g(\xi) \right| \leqslant \big(n\omega_n \epsilon^{n-1} + o\big(\epsilon^{n-1}\big)\big) \big|F_1(\epsilon) + O(\epsilon)\big| \sup_{B(x,\epsilon)} |\nabla u| \to 0.$$

Furthermore,

$$\begin{aligned}\int_{\partial B(x,\epsilon)} u(\xi) \frac{\partial F(x, \xi)}{\partial \nu_\xi}\, dS_g(\xi) &= \left(\frac{\partial F_1(\epsilon)}{\partial \epsilon} + O(1)\right) \int_{\partial B(x,\epsilon)} u(\xi)\, dS_g(\xi) \\ &= \left(\frac{1}{n\omega_n \epsilon^{n-1}} + b(\epsilon) + O(1)\right) \int_{\partial B(x,\epsilon)} u(\xi)\, dS_g(\xi) \to u(x),\end{aligned}$$

where $b(\epsilon)$ satisfies $\lim_{\epsilon \to 0} n\omega_n \epsilon^{n-1} b(\epsilon) = 0$. Altogether, we get

$$u(x) = -\int_{\partial\Omega} F(x, \xi) \frac{\partial u(\xi)}{\partial \nu_\xi}\, dS_g(\xi). \tag{2.5}$$

Since u does not vanish identically in Ω, by the above formula we get that $\frac{\partial u}{\partial \nu_\xi}|_{\partial\Omega}$ does not vanish identically on $\partial\Omega$. □

Remark 2.2. (a) When Ω is a bounded domain with $C^{2,\alpha}$-smooth boundary in a real analytic Riemannian manifold (M, g), Lemma 2.1 can be immediately proved as follows. Suppose by contradiction that $u \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ satisfies

$$\begin{cases} \triangle_g u + \lambda u = 0 & \text{in } \Omega, \\ u = 0, \quad \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Since the elliptic operator $\triangle_g + \lambda$ has real analytic coefficients in local coordinates chart, it follows from Schauder's estimate (see, for example, Theorem 6.15 of [17]) that $u \in C^{2,\alpha}(\bar{\Omega})$. Applying Holmgren's uniqueness theorem (see Theorem 2 of p. 42 in [42] or p. 433 of [46]), we obtain $u \equiv 0$ in Ω. This contradicts the assumption that u does not vanish identically in Ω.

(b) When $M = \mathbb{R}^n$, the proof of Lemma 2.1 is quite easy. Indeed, it follows from Rellich's formula for the Dirichlet eigenvalue (see [43]) that

$$\lambda = \frac{\int_{\partial\Omega} \sum_{m=1}^n (\frac{\partial u}{\partial \nu})^2 x_m \nu_m\, dS}{2 \int_\Omega u^2\, dx},$$

where $\nu(x) = (\nu_1(x), \ldots, \nu_n(x))$ with $x \in \partial\Omega$. Since $\lambda \neq 0$, we get that $\frac{\partial u}{\partial \nu}|_{\partial\Omega}$ cannot vanish identically on $\partial\Omega$.

Lemma 2.3. *Let (M, g) be an n-dimensional Riemannian manifold ($n \geqslant 2$), and let $\Omega \subset M$ be a bounded domain with C^2-smooth boundary. Then, for any τ we have*

$$W_0^{2,2}(\Omega) \cap M_\tau = \{0\},$$

where $M_\tau = \{u \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega) \mid \triangle_g u + \tau u = 0 \text{ in } \Omega\}$.

Proof. Let $v \in W_0^{2,2}(\Omega)$. It follows from Corollary 6.2.43 of [19] that

$$\frac{\partial^j v}{\partial \nu^j} = 0 \quad \text{on } \partial\Omega, \text{ for } 0 \leqslant j < \frac{3}{2}.$$

Thus, for any $v \in W_0^{2,2}(\Omega) \cap M_\tau$, we have

$$\begin{cases} \triangle_g v + \tau v = 0 & \text{in } \Omega, \\ v = 0, \quad \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

By applying Lemma 2.1, we get $v \equiv 0$ in Ω. □

Denote by σ_N (respectively, σ_D, σ_P and σ_B) the spectra of the Neumann (respectively, the Dirichlet, the clamped plate and the buckling) problem for a bounded domain in Riemannian manifold (M, g). Let

$$N^{(N)}(\tau) = \#\{\mu_k \in \sigma_N \mid \mu_k \leqslant \tau\}, \qquad N^{(D)}(\tau) = \#\{\lambda_k \in \sigma_D \mid \lambda_k \leqslant \tau\},$$
$$N^{(P)}(\tau) = \#\{\Gamma_k^2 \in \sigma_P \mid \Gamma_k \leqslant \tau\}, \qquad N^{(B)}(\tau) = \#\{\Lambda_k \in \sigma_B \mid \Lambda_k \leqslant \tau\}$$

be the counting functions of σ_N, σ_D, σ_P and σ_B, respectively. Each eigenvalue is counted as many times as its multiplicity.

Lemma 2.4. *For any τ we have*

$$N^{(N)}(\tau) = \max\left\{\dim L \,\middle|\, L \subset W^{1,2}(\Omega),\ \int_\Omega |\nabla_g u|^2 \, dV_g \leqslant \tau \int_\Omega |u|^2 \, dV_g,\ u \in L\right\}, \tag{2.6}$$

$$N^{(D)}(\tau) = \max\left\{\dim L \,\middle|\, L \subset W_0^{1,2}(\Omega),\ \int_\Omega |\nabla_g u|^2 \, dV_g \leqslant \tau \int_\Omega |u|^2 \, dV_g,\ u \in L\right\}, \tag{2.7}$$

$$N^{(B)}(\tau) = \max\left\{\dim L \,\middle|\, L \subset W_0^{2,2}(\Omega),\ \int_\Omega |\triangle_g u|^2 \, dV_g \leqslant \tau \int_\Omega |\nabla_g u|^2 \, dV_g,\ u \in L\right\}, \tag{2.8}$$

$$N^{(P)}(\tau) = \max\left\{\dim L \,\middle|\, L \subset W_0^{2,2}(\Omega),\ \int_\Omega |\triangle_g u|^2 \, dV_g \leqslant \tau^2 \int_\Omega |u|^2 \, dV_g,\ u \in L\right\}, \tag{2.9}$$

where $\nabla_g u$ is the gradient of u which has the expression in local coordinates

$$\nabla_g u = \sum_{i,j=1}^n \left(g^{ij} \frac{\partial u}{\partial x_i}\right) \frac{\partial}{\partial x_j}.$$

Proof. (i) The argument proving (2.6) and (2.7) is completely analogous to the one used in the Euclidean space (see [16] or [13]). Actually, the formulas (2.6) and (2.7) are known as Glazman's variational principle.

(ii) For any fixed τ, let $\Lambda_1, \dots, \Lambda_k$ be all the buckling eigenvalues that are not greater than τ. Then the corresponding buckling eigenfunctions $W_1, \dots, W_k$ span a k-dimensional linear subspace $\Im_k$ of $W_0^{2,2}(\Omega)$. (Suppose by contradiction that $W_m = c_1 W_1 + \dots + c_{m-1} W_{m-1} + c_{m+1} W_{m+1} + \dots + c_k W_k$ for some m, where $c_1, \dots, c_{m-1}, c_{m+1}, \dots, c_k$ are constants. Therefore, $\int_\Omega \nabla_g W_m \cdot (\nabla_g W_m - \sum_{i \neq m} c_i \nabla_g W_i) \, dV_g = 0$. Noticing that

$$\int_\Omega \nabla_g W_i \cdot \nabla_g W_j \, dV_g = \begin{cases} 1 & \text{when } i = j, \\ 0 & \text{when } i \neq j, \end{cases}$$

we obtain $\int_\Omega |\nabla_g W_m|^2 \, dV_g = 0$, so that W_m is a constant in Ω. In view of $W_m|_{\partial\Omega} = 0$ we get that $W_m \equiv 0$ in Ω, which is a contradiction.) It suffices to prove that the right-hand side of (2.8) is also k. If it is not this case, then there exists a $(k+1)$-dimensional linear subspace L_{k+1} of $W_0^{2,2}(\Omega)$ such that

$$\int_\Omega |\triangle_g u|^2 \, dV_g \leqslant \tau \int_\Omega |\nabla_g u|^2 \, dV_g \quad \text{for all } u \in L_{k+1}. \tag{2.10}$$

Thus, $E \cap L_{k+1} \neq 0$ for any linear subspace E of $W_0^{2,2}(\Omega)$ with $\operatorname{codim}(E) = k$. It follows from this and the variational formula

$$\Lambda_{k+1} = \sup_{E \subset W_0^{2,2}(\Omega),\ \mathrm{codim}\ E=k} \left(\inf_{w \in E} \frac{\int_\Omega |\triangle_g w|^2 \, dV_g}{\int_\Omega |\nabla_g u|^2 \, dV} \right)$$

that $\Lambda_{k+1} \leqslant \tau$, which is a contradiction. Therefore (2.8) holds.

(iii) The proof of (2.9) is similar to (ii). □

Lemma 2.5. *Let (M, g) be a Riemannian manifold, and let $\Omega \subset M$ be a bounded domain with C^2-smooth boundary. Suppose $\Omega_1, \ldots, \Omega_m$ are pairwise disjoint domains in Ω, each of which has piecewise C^2-smooth boundary. Arrange all the buckling eigenvalues of $\Omega_1, \ldots, \Omega_m$ in an increasing sequence*

$$\Lambda_1^* \leqslant \Lambda_2^* \leqslant \Lambda_3^* \leqslant \cdots \tag{2.11}$$

with each eigenvalue repeated according to its multiplicity, and let

$$\Lambda_1 \leqslant \Lambda_2 \leqslant \Lambda_3 \leqslant \cdots$$

be the buckling eigenvalues for Ω. Then we have for all $k = 1, 2, 3, \ldots,$

$$\Lambda_k \leqslant \Lambda_k^*. \tag{2.12}$$

Proof. For $j = 1, 2, \ldots, k$, let $\psi_j : \Omega \to \mathbb{R}^n$ be a buckling eigenfunction of Λ_j^* when restricted to the appropriate subdomain, and identically zero, otherwise. Obviously,

$$\int_\Omega \nabla_g \psi_i \cdot \nabla_g \psi_j dV_g = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Let $W_1, \ldots, W_{k-1}$ be the buckling eigenfunctions corresponding to eigenvalues $\Lambda_1, \ldots, \Lambda_{k-1}$, respectively, which satisfy

$$\int_\Omega (\nabla_g W_i \cdot \nabla_g W_j) \, dV_g = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Consider the functions f of the form

$$f = \sum_{j=1}^k \beta_j \psi_j,$$

where f satisfies

$$\sum_{j=1}^k \beta_j \int_\Omega (\nabla_g W_i \cdot \nabla_g \psi_j) \, dV_g = 0, \quad i = 1, 2, \ldots, k-1. \tag{2.13}$$

If we think of $\beta_1, \ldots, \beta_k$ as unknowns and $\int_\Omega (\nabla_g W_i \cdot \nabla_g \psi_j) \, dV_g$ as given coefficients, then system has more unknowns than equations and a nontrivial solution of (2.13) must exist. Applying Green's formula and the definition of ψ_j, we have

$$\begin{aligned} \int_\Omega (\triangle_g \psi_i)(\triangle_g \psi_j) \, dV_g &= \int_\Omega \psi_i \big(\triangle_g^2 \psi_j\big) \, dV_g = -\Lambda_j^* \int_\Omega \psi_i (\triangle_g \psi_j) \, dV_g = \Lambda_j^* \int_\Omega (\nabla_g \psi_i \cdot \nabla_g \psi_j) \, dV_g \\ &= \begin{cases} \Lambda_j^* & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \end{aligned}$$

Hence

$$\Lambda_k \int_\Omega |\nabla_g f|^2 \, dV_g \leqslant \int_\Omega |\triangle_g f|^2 \, dV_g = \sum_{j=1}^k \Lambda_j^* \beta_j^2 \leqslant \Lambda_k^* \int_\Omega |\nabla_g f|^2 \, dV_g,$$

which implies the desired result. □

3. Inequalities of eigenvalues

Proof of Theorem 1.1. (i) We shall prove $\mu_k < \lambda_k$ for all positive integer k. The proof is analogous to [14]. For any fixed τ, it follows from (2.7) of Lemma 2.4 that there exists a subspace F of $W_0^{1,2}(\Omega)$ such that $\dim F = N^{(D)}(\tau)$ and

$$\int_\Omega |\nabla_g u|^2\, dV_g \leqslant \tau \int_\Omega |u|^2\, dV_g, \quad u \in F.$$

Let $u \in F \cap M_\tau$, where $M_\tau = \{v \mid \triangle_g v + \tau v = 0 \text{ in } \Omega, \text{ and } \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega\}$. Since $\partial\Omega \in C^2$, it follows from $u \in M_\tau$ and the regularity of elliptic equations (see, for example, [1] or Theorem 8.12 of [17]) that $u \in W^{2,2}(\Omega)$. From $u \in W_0^{1,2}(\Omega)$, we get that $u = 0$ on $\partial\Omega$, as mentioned earlier. This implies that u is also a Dirichlet eigenfunction with eigenvalue τ. By Lemma 2.1, we get that $\frac{\partial u}{\partial \nu}$ cannot vanish identically in Ω, which contradicts the fact that $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$. Thus $F \oplus M_\tau$ is a direct sum and we denote it by G_τ. Let $u + v \in G_\tau \subset W^{1,2}(\Omega)$, where $u \in F$ and $v \in M_\tau$. We have

$$\begin{aligned}\int_\Omega \big|\nabla_g(u+v)\big|^2\, dV_g &= \int_\Omega \big(|\nabla_g u|^2 + |\nabla_g v|^2 + 2\nabla_g u \cdot \nabla_g v\big)\, dV_g \\ &= \int_\Omega \big(|\nabla_g u|^2 + |\nabla_g v|^2 - 2u(\triangle_g v)\big)\, dV_g \\ &\leqslant \tau \int_\Omega |u+v|^2\, dV_g,\end{aligned}$$

so that

$$N^{(N)}(\tau) \geqslant \dim G_\tau = N^{(D)}(\tau) + \dim M_\tau.$$

Taking $\tau = \lambda_k$, we have

$$\#\{\mu_j \in \sigma_N \mid \mu_j < \lambda_k\} = N^{(N)}(\lambda_k) - \dim M_{\lambda_k} \geqslant N^{(D)}(\lambda_k) = k.$$

That is, $\mu_k < \lambda_k$.

(ii) It follows from (2.9) of Lemma 2.4 that there exists a subspace H_τ of $W_0^{2,2}(\Omega)$ such that $\dim H_\tau = N^{(P)}(\tau)$ and

$$\int_\Omega |\triangle_g w|^2\, dV_g \leqslant \tau^2 \int_\Omega |w|^2\, dV_g, \quad \forall w \in H_\tau.$$

Let $K_\tau = \{v \mid \triangle_g v + \tau v = 0 \text{ in } \Omega, \text{ and } v = 0 \text{ on } \partial\Omega\}$, and let $u \in H_\tau \cap K_\tau$. Since $u \in H_\tau \subset W_0^{2,2}(\Omega)$, we find by Corollary 6.2.43 of [19] that $u = \frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$. Lemma 2.1 and $u \in K_\tau$ imply that $u = 0$ in Ω, therefore, the sum $G_\tau := H_\tau \oplus K_\tau$ is direct. Let $u = w + v \in G_\tau$, where $w \in H_\tau$, $v \in K_\tau$. It follows from Green's formula and Schwarz's inequality that for $w \neq 0$,

$$\left(\frac{\int_\Omega |\nabla_g w|^2\, dV_g}{\int_\Omega |w|^2\, dV_g}\right)^2 \leqslant \frac{\int_\Omega |\triangle_g w|^2\, dV_g}{\int_\Omega |w|^2\, dV_g}.$$

From this and the definition of H_τ, we get

$$\int_\Omega |\nabla_g w|^2\, dV_g \leqslant \tau \int_\Omega |w|^2\, dV_g.$$

Note that

$$\int_\Omega |\nabla_g v|^2\, dV_g = \tau \int_\Omega |v|^2\, dV_g, \quad \text{for } v \in K_\tau$$

and

$$\int_\Omega \nabla_g w \cdot \nabla_g v\, dV_g = -\int_\Omega w(\triangle_g v)\, dV_g = \tau \int_\Omega wv\, dV_g.$$

Therefore, for any $u = w + v \in G_\tau \subset W_0^{1,2}(\Omega)$ we have

$$\int_\Omega \left|\nabla_g(w+v)\right|^2 dV_g = \int_\Omega \left(|\nabla_g w|^2 + |\nabla_g v|^2 + 2\nabla_g w \cdot \nabla_g v\right) dV_g \leqslant \tau \int_\Omega |w+v|^2\, dV_g.$$

For $0 = w \in H_\tau$, there is equality in the above inequality. It follows that

$$N^{(D)}(\tau) \geqslant \dim G_\tau = N^{(P)}(\tau) + \dim K_\tau.$$

By taking $\tau = \Gamma_k$, we obtain

$$\#\{\lambda_j \in \sigma_D \mid \lambda_j < \Gamma_k\} = N^{(D)}(\Gamma_k) - \dim K_{\Gamma_k} \geqslant N^{(P)}(\Gamma_k) = k,$$

hence $\lambda_k < \Gamma_k$.

(iii) For fixed $\tau > 0$, (2.8) of Lemma 2.4 implies that there exists a subspace L_τ of $W_0^{2,2}(\Omega)$ such that $\dim L_\tau = N^{(B)}(\tau)$ and

$$\int_\Omega |\triangle_g w|^2\, dV_g \leqslant \tau \int_\Omega |\nabla_g w|^2\, dV_g, \quad w \in L_\tau.$$

Denote $J_\tau = \{v \mid \triangle_g^2 v - \tau^2 v = 0 \text{ in } \Omega,\ \text{and } v = \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega\}$, and put $G_\tau = L_\tau + J_\tau$. We shall prove $L_\tau \cap J_\tau = \{0\}$. Suppose that $0 \ne u \in L_\tau \cap J_\tau$. Then, in view of $u = 0$ on $\partial\Omega$ we get that $\nabla_g u$ and $\triangle_g u$ don't vanish identically in Ω. It follows from Green's formula and Schwarz's inequality that for any $u \in W_0^{2,2}(\Omega)$,

$$\int_\Omega |\nabla_g u|^2\, dV_g = \left| -\int_\Omega u(\triangle_g u)\, dV_g \right| \leqslant \left(\int_\Omega |u|^2\, dV_g\right)^{1/2} \left(\int_\Omega |\triangle_g u|^2\, dV_g\right)^{1/2}, \tag{3.1}$$

i.e.,

$$\frac{\int_\Omega |\triangle_g u|^2\, dV_g}{\int_\Omega |u|^2\, dV_g} \leqslant \left(\frac{\int_\Omega |\triangle_g u|^2\, dV_g}{\int_\Omega |\nabla_g u|^2\, dV_g}\right)^2, \quad \forall u \in W_0^{2,2}(\Omega). \tag{3.2}$$

Note that

$$\frac{\int_\Omega |\triangle_g u|^2\, dV_g}{\int_\Omega |\nabla_g u|^2\, dV_g} \leqslant \tau, \quad \forall u \in L_\tau \tag{3.3}$$

and

$$\tau^2 = \frac{\int_\Omega |\triangle_g u|^2\, dV_g}{\int_\Omega |u|^2\, dV_g}, \quad \forall u \in J_\tau. \tag{3.4}$$

Therefore, Schwarz's inequality in (3.1) is an equality, which implies that there exists a constant β such that $\triangle_g u + \beta u = 0$ in Ω. Since $u = 0$ on $\partial\Omega$, it follows that $\beta > 0$ and u is a Dirichlet eigenfunction. Thus, we find by $\frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0$ and Lemma 2.1 that $u \equiv 0$ in Ω. This shows that the sum $L_\tau \oplus J_\tau$ is direct (we still denote the direct sum by G_τ). For any $u = w + v \in G_\tau \subset W_0^{2,2}(\Omega)$, where $w \in L_\tau$, $v \in J_\tau$, we have

$$\int_\Omega \left|\triangle_g(w+v)\right|^2 dV_g = \int_\Omega \left[|\triangle_g w|^2 + |\triangle_g v|^2 + 2w\left(\triangle_g^2 v\right)\right] dV_g = \int_\Omega \left(|\triangle_g w|^2 + |\triangle_g v|^2 + 2\tau^2 wv\right) dV_g. \tag{3.5}$$

By (3.2)–(3.5), we arrive at

$$\int_\Omega \left|\triangle_g(w+v)\right|^2 dV_g \leqslant \tau^2 \int_\Omega |w+v|^2\, dV_g.$$

This implies that

$$N^{(P)}(\tau) \geqslant \dim G_\tau = N^{(B)}(\tau) + \dim J_\tau,$$

i.e.,

$$N^{(P)}(\tau) - \dim J_\tau \geqslant N^{(B)}(\tau).$$

Setting $\tau = \Lambda_k$, we see that

$$\#\left\{\Gamma_j^2 \in \sigma_P \;\middle|\; \Gamma_j < \Lambda_k\right\} = N^{(P)}(\Lambda_k) - \dim J_{\Lambda_k} \geqslant N^{(B)}(\Lambda_k) = k,$$

that is, $\Gamma_k < \Lambda_k$. $\square$

Remark 3.1. (i) For a bounded domain of a Riemannian manifold, Mazzeo [28] had showed that

$$\mu_k \leqslant \lambda_k, \quad k = 1, 2, 3, \ldots. \tag{3.6}$$

(Actually, Mazzeo proved that inequalities $\mu_{k+1} \leqslant \lambda_k$ when M is a Riemannian symmetric space of noncompact type.) Therefore, the strict inequalities

$$\mu_k < \lambda_k, \quad k = 1, 2, 3, \ldots \tag{3.7}$$

is an improvement of Mazzeo's result in the general Riemannian manifold. The following example shows that inequalities (3.7) cannot be improved as $\mu_{k+1} \leqslant \lambda_k$ for $k = 1, 2, 3, \ldots$. In fact, for the spherical cap of radius $\delta > \frac{\pi}{2}$ on the sphere of radius 1 in $\mathbb{R}^n$, one has $\mu_2(\Omega) > \lambda_1(\Omega)$ (see, Theorem 3 of p. 44 in [9]). This fact was also pointed out by Mazzeo in [28]. Therefore our strict inequalities (3.7) are sharp.

(ii) The inequalities

$$\lambda_k < \Gamma_k, \quad \text{for } k = 1, 2, 3, \ldots \tag{3.8}$$

are a generalization of Weinstein's inequality to n-dimensional Riemannian manifold. Here our proof is completely different from that of [48]. The inequalities (3.8) cannot be improved as $\lambda_{k+1} \leqslant \Gamma_k$ for $k = 1, 2, 3, \ldots$. Indeed, let Ω be the unit disk $\{x \in \mathbb{R}^2 \mid |x| < 1\}$. Denote by $J_m(r)$ the Bessel function of the first kind of order m and by $j_m^{(l)}$ its l-th positive zero. Then the Dirichlet eigenfunctions are

$$\phi_{m,l} = a_{m,l} J_m(\sqrt{\lambda_{m,l}}\, r) \begin{cases} \cos m\theta, \\ \sin m\theta, \end{cases} \quad m = 0, 1, 2, \ldots;\ l = 1, 2, 3, \ldots,$$

and the corresponding eigenvalues are $\lambda_{m,l} = (j_m^{(l)})^2$. Thus $\lambda_1(\Omega) \approx (2.4048)^2$, $\lambda_2(\Omega) = \lambda_3(\Omega) \approx (3.832)^2$. It follows from p. 231 of [40] that $\Gamma_1(\Omega) \approx (3.1962)^2$ (where 3.1962… is the first zero of $J_0(r)I_1(r) + J_1(r)I_0(r)$, $r > 0$, and $I_m(r)$ is the modified Bessel function of order m). This means that $\lambda_2(\Omega) > \Gamma_1(\Omega)$.

(iii) For $k = 2, 3, 4, \ldots$, our inequalities $\Gamma_k < \Lambda_k$ $(k = 2, 3, 4, \ldots)$ are completely new even for the case $M = \mathbb{R}^n$. It is also sharp since it cannot be further improved as $\Gamma_{k+1} \leqslant \Lambda_k$ for $k = 1, 2, 3, \ldots$. In fact, let $\Omega = \{x \in \mathbb{R}^2 \mid |x| < 1\}$. Then we claim that $\Gamma_2(\Omega) > \Lambda_1(\Omega)$. Suppose by contradiction that $\Gamma_2(\Omega) \leqslant \Lambda_1(\Omega)$. It follows from Theorem 1.1 that $\lambda_2(\Omega) < \Gamma_2(\Omega)$. Thus we get $\lambda_2(\Omega) < \Lambda_1(\Omega)$. However, for the unit disk Ω, it must be $\lambda_2(\Omega) = \Lambda_1(\Omega) \approx (3.832)^2$. This is a contradiction, and the claim is verified.

4. A counter-example to the Payne conjecture in one-dimensional case and asymptotic formula for the buckling eigenvalues in $\mathbb{R}^n$

Counter-example 4.1. First, we consider the one-dimensional buckling problem:

$$u''''(x) + \Lambda u''(x) = 0, \quad 0 \leqslant x \leqslant L, \tag{4.1}$$

$$u(0) = u(L) = 0, u'(0) = u'(L) = 0. \tag{4.2}$$

It is easy to check that the general solution of (4.1) is

$$u(x) = C_1 + C_2 x + C_3 \cos\sqrt{\Lambda}x + C_4 \sin\sqrt{\Lambda}x.$$

The boundary conditions yield the following equations for the coefficients:

$$\begin{cases} C_1 = -C_3, \quad C_2 = -\sqrt{\Lambda}C_4, \\ C_1 + C_2 L + C_3 \cos\sqrt{\Lambda}L + C_4 \sin\sqrt{\Lambda}L = 0, \\ C_2 - \sqrt{\Lambda}C_3 \sin\sqrt{\Lambda}L + \sqrt{\Lambda}C_4 \cos\sqrt{\Lambda}L = 0. \end{cases}$$

In order that this system of equations has a nontrivial solution, $\sqrt{\Lambda}$ must satisfy

$$\sin\frac{\sqrt{\Lambda}L}{2}\left[2\sin\frac{\sqrt{\Lambda}L}{2} - \sqrt{\Lambda}L\cos\frac{\sqrt{\Lambda}L}{2}\right] = 0.$$

From the equation $\sin\frac{\sqrt{\Lambda}L}{2} = 0$, we obtain that

$$\Lambda_{1,k}(L) = \left(\frac{2k\pi}{L}\right)^2, \quad k = 1, 2, 3, \ldots,$$

and the associated eigenfunctions are

$$u_{1,k}(L, x) = 1 - \cos\frac{2k\pi}{L}x, \quad k = 1, 2, 3, \ldots.$$

From the equation $2\sin\frac{\sqrt{\Lambda}L}{2} - \sqrt{\Lambda}L\cos\frac{\sqrt{\Lambda}L}{2} = 0$, we get that

$$\tan\frac{\sqrt{\Lambda}L}{2} = \frac{\sqrt{\Lambda}L}{2}. \tag{4.3}$$

If we denote by $\{\sqrt{\Lambda_{2,k}(L)} \mid k = 1,2,3,\ldots\}$ all the positive roots of (4.3), then

$$u_{2,k}(L,x) = 1 + \frac{\sqrt{\Lambda_{2,k}(L)}\sin(L\sqrt{\Lambda_{2,k}(L)}\,)}{\cos(L\sqrt{\Lambda_{2,k}(L)}\,) - 1}x - \cos\bigl(\sqrt{\Lambda_{2,k}(L)}x\bigr) - \frac{\sin(L\sqrt{\Lambda_{2,k}(L)}\,)}{\cos(L\sqrt{\Lambda_{2,k}(L)}\,) - 1}\sin\bigl(\sqrt{\Lambda_{2,k}(L)}x\bigr)$$

is the buckling eigenfunction corresponding to the eigenvalue $\Lambda_{2,k}(L)$. By solving the system of equations

$$\begin{cases} y = x, \\ y = \tan x, \end{cases}$$

we find that $\frac{2k\pi}{L} < \sqrt{\Lambda_{2,k}(L)} < \frac{(2k+1)\pi}{L}$ for all $k = 1,2,3,\ldots.$

From the above argument, we obtain all the buckling eigenvalues for the interval $[0,L]$:

$$\Lambda_1 = \left(\frac{2\pi}{L}\right)^2, \qquad \Lambda_2 = \Lambda_{2,1}(L), \qquad \Lambda_3 = \left(\frac{4\pi}{L}\right)^2, \qquad \Lambda_4 = \Lambda_{2,2}(L), \qquad \ldots. \tag{4.4}$$

A simple calculation shows that the Dirichlet eigenvalues for the interval $[0,L]$ are

$$\lambda_1 = \left(\frac{\pi}{L}\right)^2, \qquad \lambda_2 = \left(\frac{2\pi}{L}\right)^2, \qquad \lambda_3 = \left(\frac{3\pi}{L}\right)^2, \qquad \lambda_4 = \left(\frac{4\pi}{L}\right)^2, \qquad \ldots, \tag{4.5}$$

and the corresponding Dirichlet eigenfunctions are $u_k(x) = \sin(\frac{k\pi x}{L})$, $k = 1,2,3,\ldots.$ Recall that $\Lambda_{2,1}(L) < (\frac{3\pi}{L})^2$, i.e., $\Lambda_2 < \lambda_3$. This shows that the Payne conjecture is not true for the one-dimensional case.

Lemma 4.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ with C^2-smooth boundary. Then,*

$$N^{(B)}(\tau) = (2\pi)^{-n}\omega_n\bigl(vol(\Omega)\bigr)\tau^{n/2}\bigl(1 + o(1)\bigr) \quad as\ \tau \to +\infty. \tag{4.6}$$

Proof. (4.6) can be found in [18] (see also [2]). Here we give an alternative proof. For the Euclidean n-ball $B_a(x_0)$ of radius a centered at x_0, the Dirichlet eigenvalues are

$$\left(\frac{j_{l+(n/2)-1,k}}{a}\right)^2, \quad l = 0,1,2,\ldots;\ k = 1,2,\ldots,$$

where $j_{\mu,k}$ denotes the k-th positive zero of the Bessel function J_μ of the first kind of order μ. The multiplicities are $\dim\mathcal{H}_l(\mathbb{S}^{n-1})$, here $\mathcal{H}_l(\mathbb{S}^{n-1})$ is the space of spherical harmonics of degree l $(l = 0,1,2,\ldots)$. The corresponding eigenfunctions have form

$$\left[\left(\frac{j_{l+(n/2)-1}|x-x_0|}{a}\right)^{-\frac{n}{2}+1} J_{l+(n/2)-1}\left(\frac{j_{l+(n/2)-1}|x-x_0|}{a}\right)\right]\psi(\vartheta), \quad \psi \in \mathcal{H}_l\bigl(\mathbb{S}^{n-1}\bigr).$$

The buckling eigenvalues for the Euclidean n-ball of radius a are $(\frac{j_{l+(n/2),k}}{a})^2$ $(l = 0,1,2,\ldots;\ k = 1,2,\ldots)$. The multiplicities are still as stated above. In fact, any buckling eigenfunction can be unique expressed as a linear combination of a solution of the corresponding Helmholtz equation and a harmonic function (see [22]), i.e., if u is a buckling eigenfunction corresponding to Λ on the ball $B_a(x_0) \subset \mathbb{R}^n$, then

$$u = v + w \quad \text{in } B_a(x_0), \tag{4.7}$$

$$v + w = 0, \qquad \frac{\partial v}{\partial \nu} + \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial B_a(x_0), \tag{4.8}$$

where

$$v = c_1\bigl[\bigl(\Lambda^{1/2}|x-x_0|\bigr)^{-(n/2)+1} J_{l+(n/2)-1}\bigl(\Lambda^{1/2}|x-x_0|\bigr)\bigr]\psi(\vartheta),$$
$$w = c_2\bigl(\Lambda^{1/2}|x-x_0|\bigr)^l\psi(\vartheta), \qquad \psi \in \mathcal{H}_l\bigl(\mathbb{S}^{n-1}\bigr).$$

By the boundary conditions of (4.8) and identity $J'_\mu(z) = \frac{\mu}{z} J_\mu(z) - J_{\mu+1}(z)$, we get

$$-c_1\big[\Lambda^{1/2}\big(\Lambda^{1/2}|x-x_0|\big)^{-(n/2)+1} J_{l+\frac{n}{2}}\big(\Lambda^{1/2}|x-x_0|\big)\big]\big|_{|x-x_0|=a} = 0.$$

In order to let the coefficient $c_1 \neq 0$, we must have that $\Lambda^{1/2} = \frac{j_{l+\frac{n}{2},k}}{a}$ for some k.

Note that (see, for example, p. 29 of [30])

$$0 < j_{\mu,1} < j_{\mu+1,1} < j_{\mu,2} < j_{\mu+1,2} < \cdots$$

so the zeros of $J_\mu(z)$ and $J_{\mu+1}(z)$ are interlaced. This implies that

$$\lim_{\tau\to\infty} \frac{N^{(D)}_{B_a(x_0)}(\tau) - N^{(B)}_{B_a(x_0)}(\tau)}{N^{(D)}_{B_a(x_0)}(\tau)} = 0,$$

i.e.,

$$\lim_{\tau\to\infty} \frac{N^{(B)}_{B_a(x_0)}(\tau)}{N^{(D)}_{B_a(x_0)}(\tau)} = 1, \tag{4.9}$$

where $N^{(D)}_{B_a(x_0)}(\tau)$ and $N^{(B)}_{B_a(x_0)}(\tau)$ are, respectively, the counting functions of the Dirichlet and buckling eigenvalues for the ball $B_a(x_0)$.

Given $\epsilon > 0$, there exists a finite number of balls $B_1, \ldots, B_m$ satisfying $B_i \subset \Omega$ and $B_i \cap B_j \neq \emptyset$ when $i \neq j$, such that

$$(2\pi)^{-n}\omega_n\left(vol\left(\Omega - \bigcup_{i=1}^m B_i\right)\right) < \frac{\epsilon}{2}.$$

It follows from Lemma 2.5 that

$$N^{(B)}(\tau) \geqslant N^{(B)}_{B_1}(\tau) + \cdots + N^{(B)}_{B_m}(\tau).$$

By (4.9), we see that there exists an $M_1(\epsilon) > 0$ such that for $\tau > M_1(\epsilon)$,

$$N^{(B)}(\tau) \geqslant (1-\epsilon)\big(N^{(D)}_{B_1}(\tau) + \cdots + N^{(D)}_{B_m}(\tau)\big).$$

From Weyl's asymptotic formula for the Dirichlet eigenvalues, we get that there exists an $M_2(\epsilon) > 0$ such that

$$\frac{N^{(D)}_{B_i}(\tau)}{\tau^{n/2}} > (2\pi)^{-n}\omega_n\big(vol(B_i)\big) - \frac{\epsilon}{2m} \quad \text{for } \tau > M_2(\epsilon) \text{ and all } i = 1, 2, \ldots, m.$$

Taking $M(\epsilon) = \max\{M_1(\epsilon), M_2(\epsilon)\}$, we find that

$$\frac{N^{(B)}(\tau)}{\tau^{n/2}} > (1-\epsilon)\left((2\pi)^{-n}\omega_n \sum_{i=1}^m \big(vol(B_i)\big) - \frac{\epsilon}{2}\right) \quad \text{for all } \tau > M(\epsilon),$$

so that

$$\frac{N^{(B)}(\tau)}{\tau^{n/2}} > (2\pi)^{-n}\omega_n\big(vol(\Omega)\big) - \epsilon\big[(2\pi)^{-n}\omega_n\big(vol(\Omega)\big) + (1-\epsilon)\big] \quad \text{for all } \tau > M(\epsilon).$$

Hence

$$\lim_{\tau\to\infty} \frac{N^{(B)}(\tau)}{\tau^{n/2}} \geqslant (2\pi)^{-n}\omega_n\big(vol(\Omega)\big). \tag{4.10}$$

On the other hand, by applying (1.16) of Theorem 1.1 we obtain that

$$N^{(N)}(\tau) \geqslant N^{(D)}(\tau) \geqslant N^{(P)}(\tau) \geqslant N^{(B)}(\tau), \quad \text{for any } \tau.$$

It follows from Weyl's asymptotic formula for the Neumann eigenvalues in Ω that

$$\lim_{\tau\to\infty} \frac{N^{(N)}(\tau)}{\tau^{n/2}} = (2\pi)^{-n}\omega_n\big(vol(\Omega)\big), \tag{4.11}$$

so that

$$\lim_{\tau\to\infty} \frac{N^{(B)}(\tau)}{\tau^{n/2}} = \lim_{\tau\to\infty} \frac{N^{(P)}(\tau)}{\tau^{n/2}} = (2\pi)^{-n}\omega_n\big(vol(\Omega)\big). \qquad \square$$

5. Asymptotic formulas in Riemannian manifolds

Theorem 5.1. *Let (M, g) be an n-dimensional Riemannian manifold, and let $\Omega \subset M$ be a bounded domain with C^2-smooth boundary. Then,*

$$N^{(B)}(\tau) \sim (2\pi)^{-n}\omega_n\big(vol(\Omega)\big)\tau^{n/2} \quad \text{as } \tau \to +\infty, \tag{5.1}$$

$$N^{(P)}(\tau) \sim (2\pi)^{-n}\omega_n\big(vol(\Omega)\big)\tau^{n/2} \quad \text{as } \tau \to +\infty. \tag{5.2}$$

Proof. For any $x_0 \in M$, we consider a geodesic, normal coordinates system at x_0. Under the normal coordinates one can expand the metric as follows:

$$g_{ij} = \delta_{ij} - \frac{1}{3}\sum_{k,l=1}^{n} R_{iklj}x_k x_l + O\big(|x|^3\big)$$

and

$$\sqrt{\det(g_{ij})} = 1 - \frac{1}{6}\sum_{i,j=1}^{n} R_{ij}x_i x_j + O\big(|x|^3\big),$$

where R_{iklj} and R_{ij} are, respectively, the components of the curvature tensor and Ricci tensor associated with g; this is accomplished by applying the exponential map to the tangent space at 0 to obtain coordinates on a patch and then fixing things up outside (see [35], p. 59 of [10] or Chapter 10 of [8]). We let $B_{x_0}(\varrho)$ be the ball on which this coordinates system is defined. We can choose ϱ sufficiently small such that in $B_{x_0}(\varrho)$, the eigenvalues of g_{ij} and g^{ij} are between $(1+\epsilon(\varrho))^{-1}$ and $(1+\epsilon(\varrho))$, and furthermore $dV_g = \sqrt{\det(g_{ij})}\,dx$ where $(1+\epsilon(\varrho))^{-1} < \sqrt{\det(g_{ij})} < (1+\epsilon(\varrho))$. Here $\epsilon(\varrho)$ is a positive function of variable ϱ, and $\epsilon(\varrho) \to 0$ as $\varrho \to 0$. Let N be any compact sub-manifold in M with

$$Rc_g \geqslant -c \quad \text{on } N, \tag{5.3}$$

where c is a positive constant. The classical Bochner–Lichnerowicz–Weitzenböck formula (see [26]) reads

$$\int_N \big|\nabla_g^2 u\big|^2\,dV_g = \int_N |\triangle_g u|^2\,dV_g - \int_N Rc_g(\nabla_g u, \nabla_g u)\,dV_g \quad \text{for any } u \in C_0^\infty(N),$$

where $|\nabla_g^2 u|^2$ is defined in an invariant ways as

$$\big|\nabla_g^2 u\big|^2 = \nabla^l\nabla^k u\nabla_l\nabla_k u = g^{pl}g^{kq}\left(\frac{\partial^2 u}{\partial x_k\partial x_l} - \Gamma_{kl}^m\frac{\partial u}{\partial x_m}\right)\left(\frac{\partial^2 u}{\partial x_p\partial x_q} - \Gamma_{pq}^r\frac{\partial u}{\partial x_r}\right)$$

Together with (5.3), it implies that

$$\int_N \big|\nabla_g^2 u\big|^2\,dV_g \leqslant \int_N |\triangle_g u|^2\,dV_g + c\int_N |\nabla_g u|^2\,dV_g.$$

Denote by $\mathbb{B}_0(\varrho)$ the ball of $\mathbb{R}^n$ with the center 0 and radius $\varrho > 0$, and denote by $\triangle u$ and ∇u the usual the Laplacian and gradient of u in $\mathbb{R}^n$. Passing in the coordinates system, we find by a similar way as in p. 135 of [12] that

$$(\triangle_g u)^2 \leqslant (\triangle u)^2 + \tilde{\epsilon}(\varrho)\big|\nabla^2 u\big|^2 + \tilde{\epsilon}(\varrho)|\nabla u|^2, \quad \text{for } u \in C_0^2\big(\mathbb{B}_0(\varrho)\big)$$

where $\tilde{\epsilon}(\varrho) \to 0$ as $\varrho \to 0$, while by the Bochner–Lichnerowicz–Weitzenböck formula,

$$\int_{\mathbb{B}_0(\varrho)} \big|\nabla^2 u\big|^2\,dx = \int_{\mathbb{B}_0(\varrho)} (\triangle u)^2\,dx.$$

Note that for any $u \in C_0^2(\mathbb{B}_0(\varrho))$,

$$\begin{aligned}\int_{B_{x_0}(\varrho)} |\nabla_g u|^2\,dV_g &= \int_{\mathbb{B}_0(\varrho)} \sum_{i,j=1}^{n} g^{ij}\sqrt{\det(g_{ij})}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\,dx \\ &\geqslant \int_{\mathbb{B}_0(\varrho)} \big(1+\epsilon(\varrho)\big)^{-2}|\nabla u|^2\,dx.\end{aligned}$$

Thus, we have that for any $u \in C_0^2(\mathbb{B}_0(\varrho))$,

$$\frac{\int_{B_{x_0}(\varrho)} (\triangle_g u)^2\, dV_g}{\int_{B_{x_0}(\varrho)} |\nabla_g u|^2\, dV_g} \leqslant \big(1+\epsilon(\varrho)\big)^2\big(1+\tilde{\epsilon}(\varrho)\big)\frac{\int_{\mathbb{B}_0(\varrho)} (\triangle u)^2\, dx}{\int_{\mathbb{B}_0(\varrho)} |\nabla u|^2\, dx} + \big(1+\epsilon(\varrho)\big)^2\tilde{\epsilon}(\varrho). \tag{5.4}$$

We may always assume that ϱ is small enough such that $\lambda_1(\mathbb{B}_0(\varrho)) > 1$, where $\lambda_1(\mathbb{B}_0(\varrho))$ is the first Dirichlet eigenvalue for $\mathbb{B}_0(\varrho)$. Since the geodesic open balls $\{B_{x_0}(\varrho) | x_0 \in M\}$ cover $\bar{\Omega}$, it follows from Lebesgue's lemma (see, for example, Theorem 6.27 of [41]) that there exists a constant $\gamma > 0$ such that if any subdomain $G \subset \bar{\Omega}$ satisfies $diam(G) < \gamma$, then G must be contained in some $B_{x_0}(\varrho)$. Let us divide the domain Ω into h subdomains $G_1, G_2, \dots, G_h$ with piecewise C^2-smooth boundaries such that $diam(G_j) < \gamma$, $1 \leqslant j \leqslant h$. It follows from Lemma 2.5 that the k-th buckling eigenvalue Λ_k for the domain Ω is not greater than the k-th number Λ_k^* in the sequence consisting of all the buckling eigenvalues of the subdomains G_j (arranged according to increasing magnitude and taken with their respective multiplicity). Thus, we have

$$N^{(B)}(\tau) \geqslant N_{G_1}^{(B)}(\tau) + N_{G_2}^{(B)}(\tau) + \cdots + N_{G_h}^{(B)}(\tau), \tag{5.5}$$

where $N^{(B)}(\tau)$ and $N_{G_j}^{(B)}(\tau)$ are the numbers of the buckling eigenvalues less than or equal to τ for Ω and G_j, respectively. For each subdomain G_j, we take a point $p_j \in G_j$ such that $G_j' \subset \mathbb{B}_0(\varrho)$, where $G_j' = \{x' \in \mathbb{R}^n \mid x' = \mathrm{Exp}_{p_j}^{-1} x,\ x \in G_j\}$. Therefore, under normal coordinates at p_j, the inequality (5.4) holds for any $u \in W_0^{2,2}(G_j')$. This implies that

$$\Lambda_k(G_j) \leqslant \big(1+\epsilon(\varrho)\big)^2\big(1+\tilde{\epsilon}(\varrho)\big)\Lambda_k\big(G_j'\big) + \big(1+\epsilon(\varrho)\big)^2\tilde{\epsilon}(\varrho), \quad k = 1,2,3,\dots. \tag{5.6}$$

By Theorem 1.1 and the Faber–Krahn inequality (see, for example, Theorem 2 of p. 87 in [9]), we have

$$1 \leqslant \lambda_1\big(\mathbb{B}_0(\varrho)\big) \leqslant \lambda_1\big(G_j'\big) \leqslant \Lambda_1\big(G_j'\big) \leqslant \Lambda_k\big(G_j'\big), \quad 1 \leqslant j \leqslant h.$$

It follows from this and (5.6) that

$$\Lambda_k(G_j) \leqslant \big(1+\epsilon(\varrho)\big)^2\big(1+2\tilde{\epsilon}(\varrho)\big)\Lambda_k\big(G_j'\big), \quad j = 1,2,\dots,h,\ k = 1,2,3,\dots,$$

so that

$$N_{G_j}^{(B)}(\tau) \geqslant N_{G_j'}^{(B)}\left(\frac{\tau}{(1+\epsilon(\varrho))^2(1+2\tilde{\epsilon}(\varrho))}\right), \quad j = 1,2,\dots,h. \tag{5.7}$$

By Theorem 4.2, we have that

$$N_{G_j'}^{(B)}\left(\frac{\tau}{(1+\epsilon(\varrho))^2(1+2\tilde{\epsilon}(\varrho))}\right) = (2\pi)^{-n}\omega_n\big(vol\big(G_j'\big)\big)\left(\frac{\tau}{(1+\epsilon(\varrho))^2(1+2\tilde{\epsilon}(\varrho))}\right)^{n/2}\big(1+o(1)\big)$$
$$\text{as } \tau \to \infty. \tag{5.8}$$

It follows from (5.5), (5.7) and (5.8) that, as $\tau \to +\infty$,

$$N^{(B)}(\tau) \geqslant (2\pi)^{-n}\omega_n \sum_{j=1}^{h}\big(vol\big(G_j'\big)\big)\left(\frac{\tau}{(1+\epsilon(\varrho))^2(1+2\tilde{\epsilon}(\varrho))}\right)^{n/2}\big(1+o(1)\big).$$

Recall that $dV_g = \sqrt{\det(g_{ij})}\,dx$ with $(1+\epsilon(\varrho))^{-1} < \sqrt{\det(g_{ij})} < (1+\epsilon(\varrho))$. We have

$$\big(1+\epsilon(\varrho)\big)^{-1}\big(vol\big(G_j'\big)\big) < vol(G_j) < \big(1+\epsilon(\varrho)\big)\big(vol\big(G_j'\big)\big),$$

so that

$$\sum_{j=1}^{h}\big(vol\big(G_j'\big)\big) \geqslant \big(1+\epsilon(\varrho)\big)^{-1}\sum_{j=1}^{h} vol(G_j) = \big(1+\epsilon(\varrho)\big)^{-1}\big(vol(\Omega)\big).$$

This implies that

$$N^{(B)}(\tau) \geqslant (2\pi)^{-n}\omega_n\big(1+\epsilon(\varrho)\big)^{-1}\big(vol(\Omega)\big)\left(\frac{\tau}{(1+\epsilon(\varrho))^2(1+2\tilde{\epsilon}(\varrho))}\right)^{n/2}\big(1+o(1)\big) \quad \text{as } \tau \to \infty. \tag{5.9}$$

Hence

$$\varliminf_{\tau\to\infty}\frac{N^{(B)}(\tau)}{\tau^{n/2}} \geqslant (2\pi)^{-n}\omega_n\big(vol(\Omega)\big). \tag{5.10}$$

For, we may choose the quantity ϱ arbitrarily, and by taking a sufficiently small fixed ϱ, make the factor of $\tau^{n/2}$ in (5.9) arbitrarily close to $(2\pi)^{-n}\omega_n(vol(\Omega))$ for sufficiently large τ.

On the other hand, it follows from (6) of [29] that, for the bounded domain Ω in Riemannian manifold (M,g),

$$\sum_{k=1}^{\infty} e^{-t\mu_k} = (4\pi t)^{-n/2}\left[vol(\Omega) + \frac{1}{4}\sqrt{4\pi t}\big(vol(\partial\Omega)\big) + \frac{t}{3}\int_{\Omega} R\,dV_g - \frac{t}{6}\int_{\partial\Omega} J\,dS_g + o\big(t^{3/2}\big)\right], \tag{5.11}$$

where R is the scalar curvature at a point of M, and J the mean curvature at a point of $\partial\Omega$. From (5.11), we have

$$\int_{0-}^{\infty} e^{-t\tau}\,dN^{(N)}(\tau) = \sum_{k=1}^{\infty} e^{-t\mu_k} \sim (4\pi t)^{-n/2}\big(vol(\Omega)\big), \quad \text{as } t \to 0,$$

where $N^{(N)}(\tau)$ is the number of the Neumann eigenvalues less than or equal to τ for Ω, and $\int_{0-}^{\infty} e^{-t\tau}\,dN^{(N)}(\tau)$ is the Riemann–Stieltjes integral on $[0,+\infty)$ (Note that $\int_{0-}^{\infty} e^{-t\tau}\,dN^{(N)}(\tau)$ means $\lim_{\delta\to 0+}\int_{-\delta}^{\infty} e^{-t\tau}\,dN^{(N)}(\tau)$.) It follows from Proposition 3.2 of p. 89 in [47] (or Theorem 15.3 of [24]) that

$$N^{(N)}(\tau) \sim (2\pi)^{-n}\omega_n\big(vol(\Omega)\big)\tau^{n/2}, \quad \text{as } \tau \to \infty,$$

i.e.,

$$\lim_{\tau\to\infty} \frac{N^{(N)}(\tau)}{\tau^{n/2}} = (2\pi)^{-n}\omega_n\big(vol(\Omega)\big). \tag{5.12}$$

By (1.16) of Theorem 1.1, we have that

$$N^{(N)}(\tau) \geqslant N^{(D)}(\tau) \geqslant N^{(P)}(\tau) \geqslant N^{(B)}(\tau), \quad \text{for any } \tau. \tag{5.13}$$

It follows from (5.10), (5.12) and (5.13) that

$$\lim_{\tau\to\infty} \frac{N^{(B)}(\tau)}{\tau^{n/2}} = \lim_{\tau\to\infty} \frac{N^{(P)}(\tau)}{\tau^{n/2}} = \lim_{\tau\to\infty} \frac{N^{(D)}(\tau)}{\tau^{n/2}} = \lim_{\tau\to\infty} \frac{N^{(N)}(\tau)}{\tau^{n/2}} = (2\pi)^{-n}\omega_n\big(vol(\Omega)\big). \quad \square \tag{5.14}$$

Remark 5.2. (i) For the Dirichlet and Neumann eigenvalue problems, Seeley [44] and Pham [36] showed that if $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^∞-smooth boundary, then the following sharp remainder estimate holds:

$$N^{(D)}(\tau) = (2\pi)^{-n}\omega_n\big(vol(\Omega)\big)\tau^{n/2}\big(1 + O\big(\tau^{-\frac{1}{2}}\big)\big), \quad \text{as } \tau \to \infty.$$

In [44], Seeley used the method of hyperbolic equations which is the most precise of the known Tauberian methods. Seeley in [45] has generalized the above result to n-dimensional Riemannian manifolds.

(ii) For the Dirichlet and Neumann eigenvalues of a bounded domain Ω in a smooth Riemannian manifold M, Ivrii (see [21]) has established:

$$N^{\pm}(\tau) = (2\pi)^{-n}\omega_n\big(vol(\Omega)\big)\tau^{n/2} \pm \frac{1}{4}(2\pi)^{-n+1}\omega_{n-1}\big(vol(\partial\Omega)\big)\tau^{(n-1)/2} + o\big(\tau^{(n-1)/2}\big), \quad \text{as } \tau \to +\infty, \tag{5.15}$$

under an additional assumption (roughly, that the set of "multiply reflected periodic geodesics in $\bar{\Omega}$ is of measure zero"), where $N^+(\tau)$ and $N^-(\tau)$ denote the counting functions of σ_N and σ_D, respectively. Melrose [32] independently obtained the same asymptotic estimate (5.15) for Riemannian manifolds with concave boundary. However, Ivrii's method is no longer valid for the buckling eigenvalues.

Proof of Theorem 1.2. Taking $\tau = \Lambda_k$ (respectively, $\tau = \Gamma_k$) in Theorem 5.1, we immediately obtain the asymptotic formula (1.19) (respectively, (1.20)) of the theorem. $\square$

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